

Octonions

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Abstract

In this paper we review the topological model for the quaternions based upon the Dirac string trick. We then extend this model, to create a model for the octonions - the non-associative generalization of the quaternions.

1 Introduction

In this paper we give a topological/combinatorial model of the octonions that is an extension of an already-existing model of the quaternions. The model of the quaternions [1] is based on the belt trick or Dirac string trick, and will be reviewed in the first section of this paper. We then proceed to find a corresponding model for the octonions. The octonions are a non-associative generalization of the quaternions that have been used directly and speculatively in physics for some time [2, 3, 5, 4, 7]. We hope that the present model will lead to new physical insight.

Recall first the definition of the the quaternions. The quaternions are an associative algebra over the real numbers generated by linearly independent elements

$$1, i, j, k$$

with

$$i^2 = j^2 = k^2 = ijk = -1.$$

From these equations, it follows that $ij = k, jk = i, ki = j$ and $ji = -k, kj = -i, ik = -j$. In general, a quaternion A has the form

$$A = a + bi + cj + dk$$

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where a, b, c, d are real numbers. The *conjugate* \bar{A} of A is defined by the equation

$$\bar{A} = a - bi - cj - dk$$

and has the property that

$$A\bar{A} = a^2 + b^2 + c^2 + d^2,$$

showing that non-zero quaternions have multiplicative inverses.

The octonions are a non-associative algebra obtained by adding a new element L to the quaternions. If A, B, C, D are quaternions, then the products in the octonions are all determined by the following formula

$$(A + LB)(C + LD) = (AC - D\bar{B}) + L(CB + \bar{A}D).$$

Another way of putting this, that is useful for our purposes is the following: Suppose that

$$x, y \in \{i, j, k\}.$$

Then

$$LL = -1,$$

$$xL = -Lx,$$

$$(Lx)y = L(yx),$$

$$(Lx)(Ly) = yx,$$

$$x(Ly) = -L(xy).$$

We will use this form of the identities for octonion multiplication to check the properties of our model of the octonions.

2 The Quaternions

Through a topological property, commonly referred to as the *Dirac string trick*, we can construct a physical/topological model of the quaternion group. In this model one takes a geometric object in Euclidean three-dimensional space and attaches a belt (i.e. a space homeomorphic to the cross product of a unit interval with itself) to the object and to a reference point. The reference point is often taken to be the north pole of a two dimensional sphere surrounding the object. Rotations of the object carry the belt along, twisting it without causing self-intersections or singularities. A 2π rotation of the object about an axis causes a twist to appear in the belt that cannot be removed by topological isotopy of the belt, leaving the endpoints of the belt fixed to the object and to the reference point. But a 4π rotation gives a state of the belt that can be isotoped to its original (untwisted) state by isotopy fixing the endpoints. This topological fact is usually called the belt trick or Dirac string trick.

If we attach a belt to an object O with symmetry group G in $SO(3)$ ((the group of orientation preserving rotations of Euclidean three-space), then each symmetry of the object acquires two possible states: The two states differ by a 2π twist of the belt. The result is a doubling of the symmetry group of the object to a new group \hat{G} . If we take the object to be a *belt buckle* (i.e. a rectangle), then the symmetry group is $G = Z/2Z \times Z/2Z$ of order 4, and \hat{G} is the eight-element quaternion group.

Previously, this property has been used to construct a model for the quaternions [1] by working specifically with belt and belt-buckle. The reason behind the working of such models has to do with the fact that one has a double covering $p : SU(2) \rightarrow SO(3)$ where $SU(2)$ denotes the unitary 2×2 matrices of determinant one. $SU(2)$ is isomorphic with the quaternions of unit length, and one can show that the group \hat{G} is isomorphic with the inverse image of G under this double covering mapping. Thus $\hat{G} = p^{-1}(G)$. The topology of the belt returning to its identity state from a 4π rotation is a consequence of the fact that the fundamental group of the Lie group $SO(3)$ is of order two. The relationship with $SU(2)$ demonstrates clearly the relation between the quaternions and the physical situations in which they arise in naturally. In particular, the non-triviality of the 2π rotation and the triviality of the 4π rotation corresponds to the fact in quantum mechanics that the wave function of a fermion changes by a sign if the system undergoes a rotational symmetry of 2π . Symmetries of the system are mapped to unitary transformations in quantum mechanics, and so correspond to lifts to the double cover of $SO(3)$.

To construct this model of the quaternions using belt and buckle, we consider a belt that has been fixed to a wall with the non-buckle end. We consider π rotations of the belt buckle about the three standard cartesian axes which we correspond to the three quaternionic roots of -1 : i, j , and k . We then get that carrying out j after i yields the same result as performing k - likewise for any other combination of i, j and k - with $-x$ equivalent to x but with the twisting of the belt in the opposite direction. We also get that carrying out any operation twice yields a belt that is twisted around by a full 2π , which we then call -1 . The final step in getting our model is to recall the Dirac belt trick which tells us that if we perform -1 twice - giving us a 4π rotation - we can remove all of the twisting without rotating the belt buckle. We use this operation to remove any extraneous twisting, and find that our operations exactly correspond to all of the elements of the quaternions. We note that the operations are performed from left to right along a string of elements.

We can observe that beginning with i and performing j we reach k . And that beginning with j and performing i we instead reach $-k$. In this way the quaternions are realized by the behaviour of a belt attached to a wall in three dimensional space. See [1] for a more detailed discussion of the belt trick and its relationship with the quaternions.

3 The Octonions

We construct our model for the octonions in a similar manner to the model for the quaternions. Rather than using a belt, we will instead use a two toned

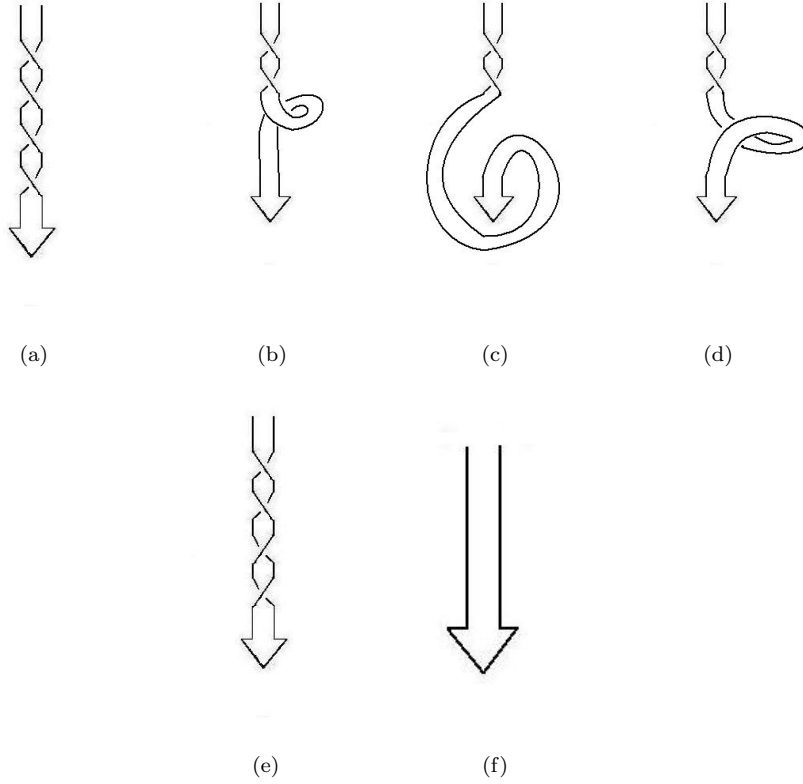


Figure 1: Step-by-step demonstration of the Dirac string trick on a belt

ribbon (black on the back, and white on the front) with an arrowhead attached to one end (much as our belt had a buckle). The other end is then attached to the interior of a ring (much as our belt was attached to a wall). Lastly on the side of the ring we affix a flag that allows us to keep track of the orientation of the ring. We will describe all operations with respect to viewing the ring from above the plane in which it lies with the white side of the ribbon facing us at the point where it is attached to the ring. Additionally, as in the model of the quaternions, we will perform all composite operations from left to right. We then begin with the identity as shown in figure 5.

We can also then bring with us from our model of the quaternions i, j and k as shown in figure 6.

Our operation L is defined by moving the flag to the opposite side of the ring and our operations Li , Lj and Lk are defined by corresponding *rotations of the external hoop while holding the ribbon stationary*, together with a reversal of the colour of the ribbon. Note that rotating the external hoop will create twists in the ribbon. After performing each operation we perform global rotations to return our setup to standard form - the white side of the ribbon attached to the top of the hoop. No extra twists are created by these global rotations.

We then have the form of L , Li , Lj and Lk applied to the identity given by figure 7.

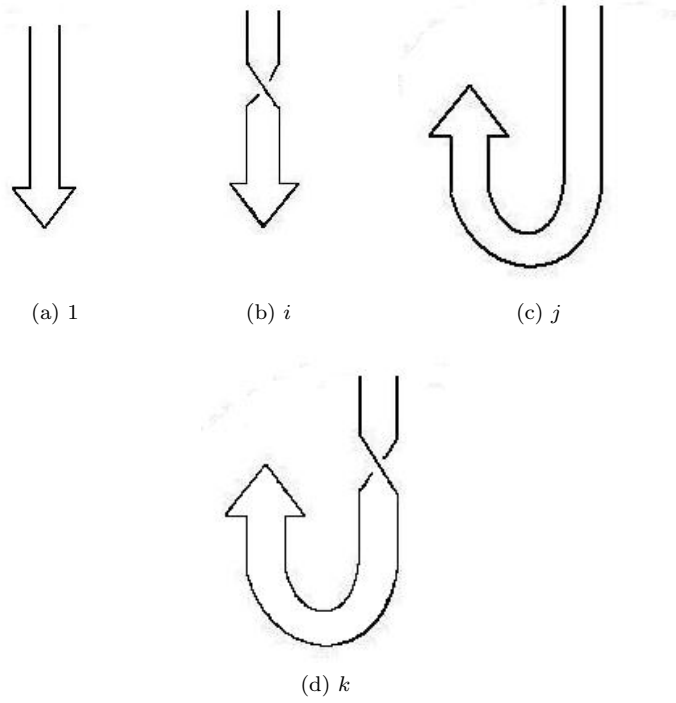


Figure 2: Belt states of the identity and the quaternions i , j and k

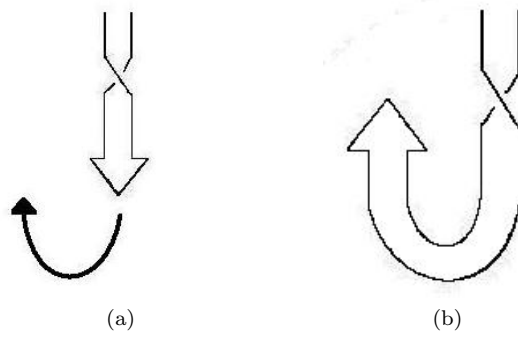


Figure 3: $ij = k$

This gives us the basic form of the operations. The issue is that some ambiguity exists in how to apply each operation given different configurations of the flag, colours and locations of the arrow.

4 The Rules

To introduce the consistent rules we will introduce some terminology. We will refer to an arrowhead *pointing up* if we have a configuration such as that for

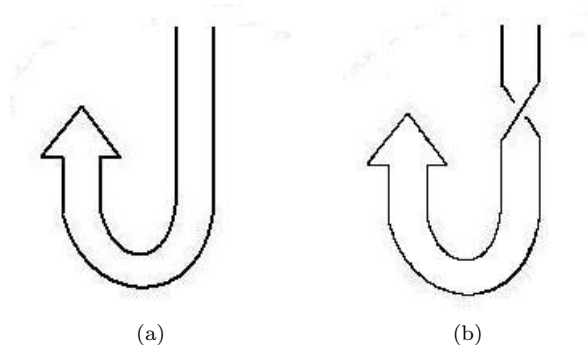


Figure 4: $ji = -k$

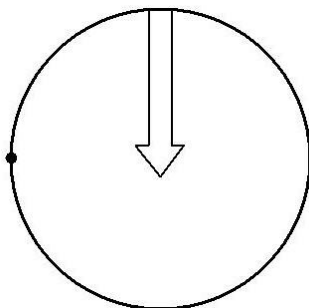


Figure 5: Identity

j (see figure 6) where the arrowhead points towards the top half of the hoop. We will call a state *flag-right* if when viewed in standard form the flag is on the right side of the hoop. Lastly, we will say a state has a *black arrowhead* if there is a twist in the ribbon in such a manner that the ribbon attaches to the arrowhead with the black side of the ribbon facing up.

4.1 The Quaternion Rules

1. The operation i is a clockwise rotation of the arrowhead with respect to the hoop (through an axis through the arrowhead to the top of the hoop). *This direction of rotation is reversed if the state is flag-right or if the arrowhead is pointing up, but not for both.*
2. The operation j is a clockwise rotation of the arrowhead as viewed from above the hoop for any configuration of state.
3. The operation k is performed by flipping the arrowhead under the ribbon, *unless the arrowhead is pointing up* - in which case we instead flip the arrowhead over the ribbon.

It is important to note that excluding the reversal of i for a state being flag-right that these are the standard rules for the model of the quaternions.

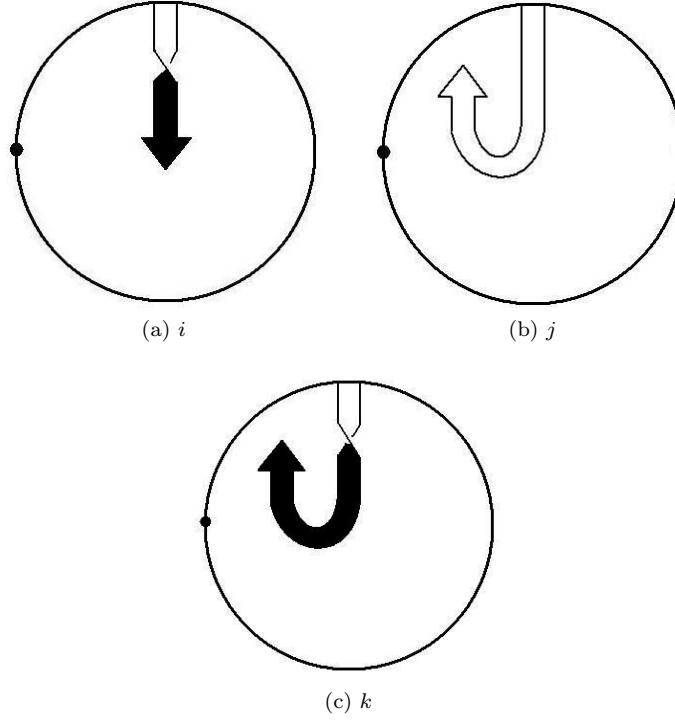


Figure 6: i , j and k

4.2 Rules for L and the other Octonions

1. The operation L is defined by switching the side of the hoop that the flag is attached to, and *performing a full 2π rotation of the hoop (or - alternately - the arrowhead) if the arrowhead is pointing up or if the state is flag-right, but not for both.*
2. Li takes the form - when looking through from the base of the hoop to the root of the ribbon - of rotating the hoop clockwise and reversing the colouring of the ribbon. If the arrowhead is pointing up, we instead rotate counterclockwise. Looking to figure 8 we see this process at each step: from the identity we first rotate the hoop clockwise (figure 8b), then we reverse the colour of the ribbon (figure 8c) and then - as we are already now in standard form, with the hoop showing the attachment to the white ribbon - we are finished.
3. Lj takes the form of rotating the hoop clockwise - from the standard view - and reversing the colouring of the ribbon. If the state has a black arrowhead, or is flag-right we reverse the direction of rotation, but not if the state is both. Taking Lj step by step we get (in figure 9): a rotation of the hoop clockwise in the plane (figure 9b), reversing the colouring (figure 9c) and then lastly performing a rotation of the total system to put it into standard form (figure 9d).

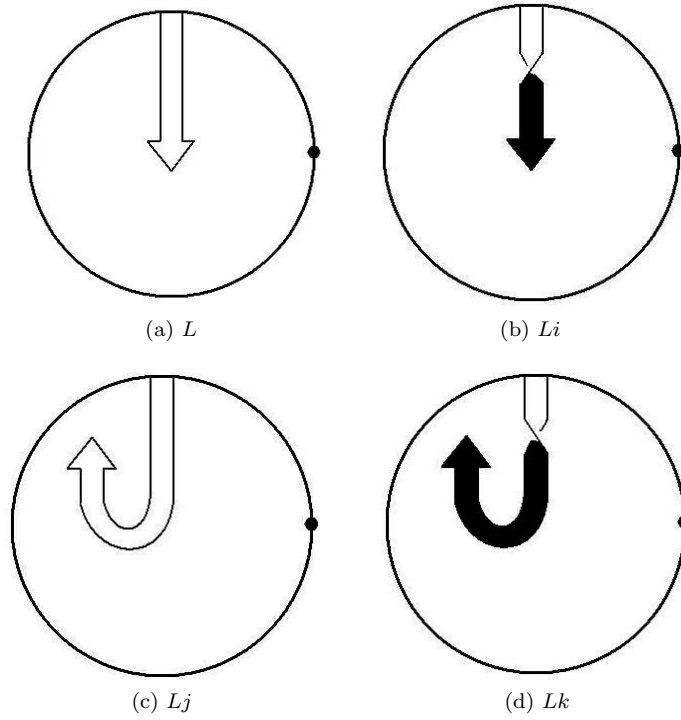


Figure 7: L, Li, Lj and Lk

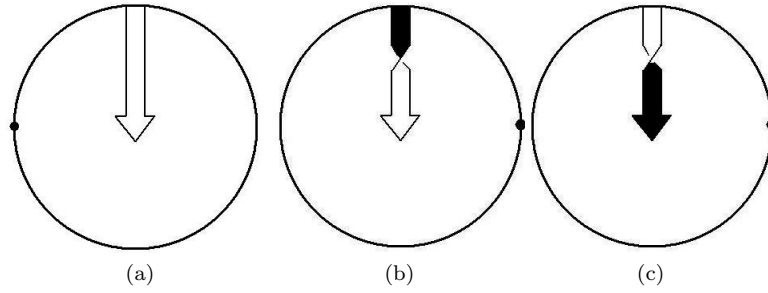


Figure 8: Step by step application of Li

4. Lk takes the form of a clockwise rotation of the hoop - when viewing the hoop from the left - and reversing the colouring of the ribbon. We reverse the direction of rotation for each of a black arrowhead, flag-right or an arrowhead pointing up (i.e. we would then have a clockwise rotation for a state with any two, and a counterclockwise rotation for a state that was all three). The step by step process of applying Lk to the identity is given by: rotating the hoop (figure 10b), changing the colour (figure 10c), and then rotating the hoop and ribbon together to put it into standard form

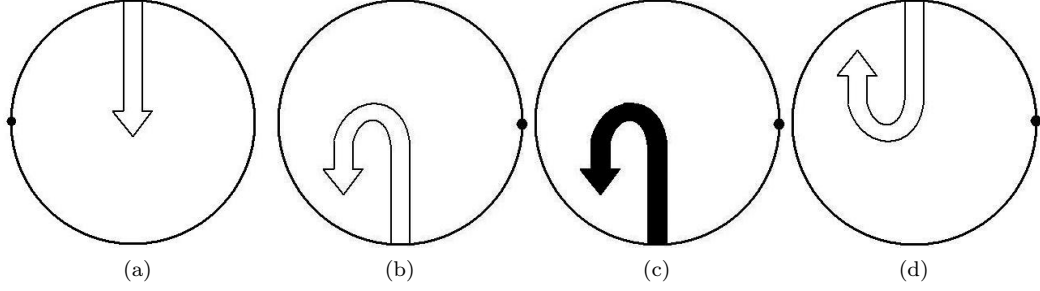


Figure 9: Step by step application of Lj

(figure 10d).

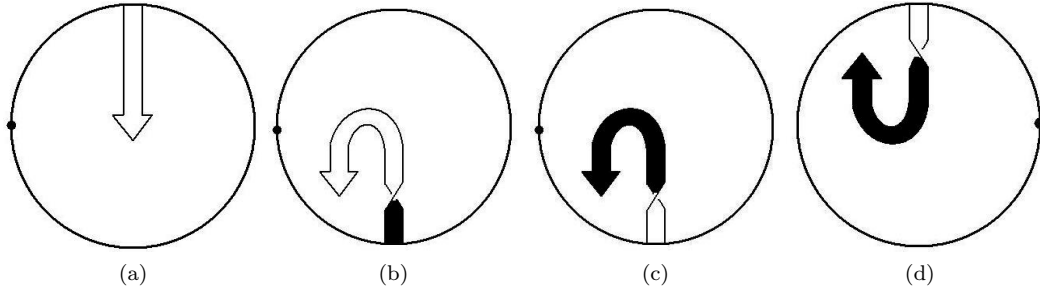


Figure 10: Step by step application of Lk

To demonstrate these rules in practice (not just multiplied to the identity) and simultaneously to demonstrate that we have captured the non-associativity of the octonions, we'll perform two example calculations: $(Lj)k$ and $j(Li)$. To perform the first (see figure 11), we begin with Lj as performed on the identity, and then we carry out k - as Lj has an upward pointing arrow, the direction of k 's rotation is reversed from its application on the identity, the result - after we remove deformation - is $(Lj)k = L(kj) = L(-i) = -Li$, comparing this to $L(jk) = Li$ and we find that we've captured the non-associativity in this scenario. Next, we'll take $j(Li)$ (see figure 12), here we begin by performing j , and then perform Li which - as the arrowhead is pointing up - is a counterclockwise rotation of the ribbon, followed by a reversal of colouring (see figure 12c). We now have the arrowhead to the right of the ribbon, we can drag the ribbon to the other side of the arrowhead (this reverses the crossing) and we arrive at $-Lk$ which as $j(Li) = L(ji) = -L(k)$ is as desired.

5 Conclusion

We have now given an assignment of an operation to each of the octonions acting upon a state of our belt-hoop apparatus. To demonstrate that this is indeed a

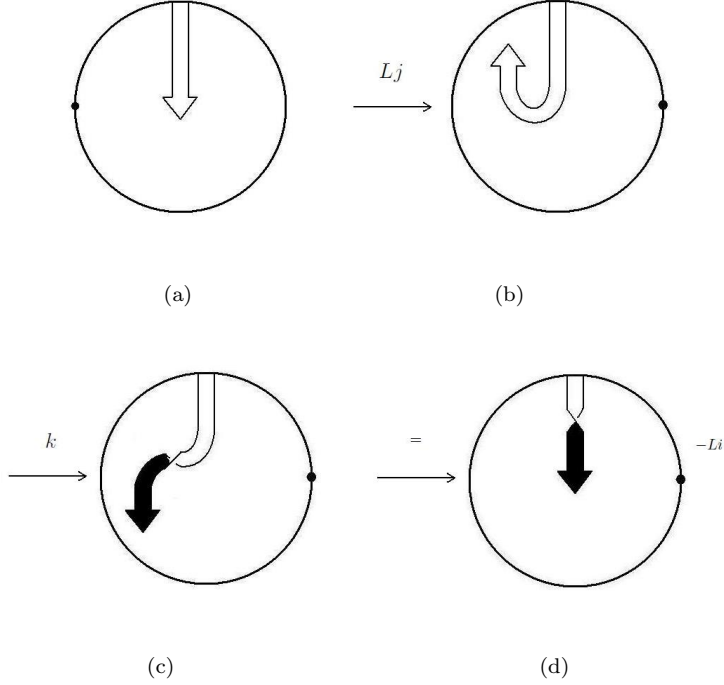


Figure 11: Calculation of $(Lj)k$

model of the octonions, one only needs to check that any string of elements of the octonions can be equivalently resolved by the standard rules of octonionic multiplication or by applying them on our state in order - we stress here that the operations should be performed from left to right (i.e. Li is L followed by i).

This model of the octonions introduces several questions. The original belt model of the quaternions is strongly related to the quaternions being a representation of $SU(2)$, and $SU(2)$ being a double cover of the rotation group $SO(3)$. The fact that this model of the octonions is an extension of the quaternionic model leads to the question of whether an analogue to the relationship with $SU(2)$ and $SO(3)$ exists. In particular the form of L - in some sense resembling a parity operation - plays to these speculations.

Extending on the relationship of the quaternions with $SU(2)$ is the question of whether this model could provide illumination to attempts to use the octonions to construct the standard model of particle physics - such as the attempt in [2]. Here again the resemblance of L to parity inversion is suggestive of something more profound. We will continue these considerations in a sequel to the present paper.

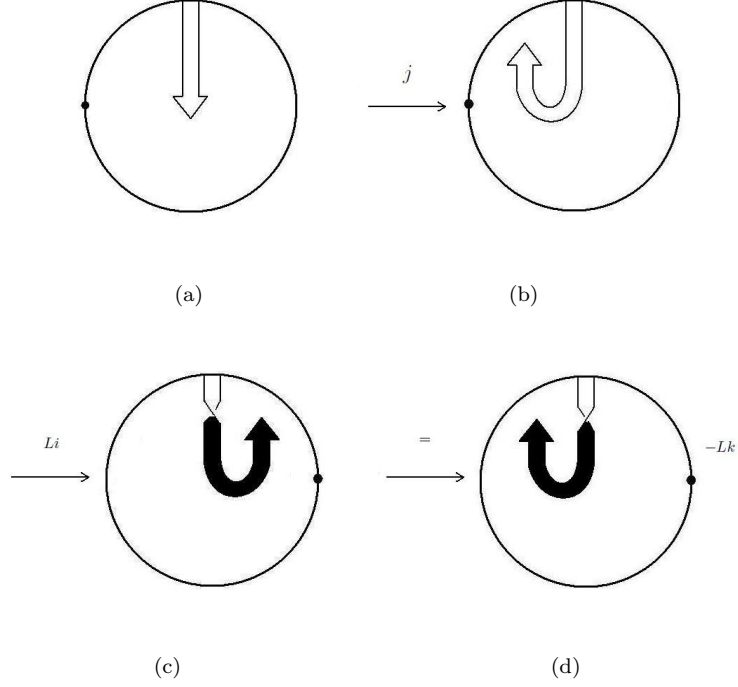


Figure 12: Calculation of $j(Li)$

6 Acknowledgements

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References

- [1] L. H. Kauffman, “Knots and Physics,” World Scientific Pub. Co. (1991,1993,2001) p. 422-423.
- [2] C. Furey, “Unified Theory of Ideals,” arXiv:1002.1497 [physics.gen-ph].
- [3] S. Catto, ”Algebraic Realization of Quark-Diquark Supersymmetry,” hep-th/9811069.
- [4] J. Baez, “The Octonions”. Bull. Amer. Math. Soc. 39 (2002), 145-205.
- [5] F. Gursey and C.-H. Tze, “On the Role of Division, Jordan and Related Algebras in Particle Physics,” World Scientific, Singapore (1996).

- [6] J. Conway and D. Smith, “On Quaternions and Octonions: Their Geometry, Arithmetic, and Symmetry,” A. K. Peters Pub. Co. (2003).
- [7] S. Okubo, “Introduction to Octonion and Other Non-Associative Algebras in Physics,” Cambridge University Press (1995)